Some applications of the quantum Weyl groups

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Abstract. *The quantum Weyl groups were introduced by the second author some years ago by using an idea of V.G. Drinfeld. We now apply this notion to obtain a formula for the universal quantum R-matrix of a simple Lie algebra.*

We also obtain some results in connection with Hecke algebras.

0.1. V.G. Drinfeld introduced the notion of a quantum group in [2]. The most investigated class of quantum groups is the class of quantized universal enveloping algebras of simple Lie algebras. Many notions of the usual theory of simple Lie algebras have quantum counterparts. In this works we shall deal with a quantum analogue of the Weyl group. This analogue was introduced by the second author in [15] by means of an idea of V.G. Drinfeld.

We would like to remark that our approach is different from that in some papers by G. Lusztig [8], [11]. Namely we define quantum Weyl group as Hopf algebra (this is important). We are then able to recover Lusztig's automorphisms $\{T_i\}$ as $\{Ad_{\bar{s}_i^{-1}}\}$ where $\{\bar{s}_i\}$ is our quantum analogue of simple reflections.

These $\{\bar{s}_i\}$ were introduced first in [15].

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0.2. Let us discuss the motivations. Let $\mathscr{G}$ be a simple complex Lie algebra and G a corresponding Lie group. We may consider the Weyl group W_1 as the group generated by simple reflections in $\mathscr{H}^*$ where $\mathscr{H}$ are the Cartan subalgebra in $\mathscr{G}$. At the same time we can define $W_2 = N(T)/T$ where T is the maximal torus.

It is known that $W_1 \simeq W_2$, but we will see that for their quantum counterparts this is not the case. Roughly speaking the quantum analogue of W_1 is W_1 itself, but the quantum analogue of W_2 is what we call quantum Weyl group. The main observation is as follows.

It is known that one can choose a subgroup $\tilde{W}_2 \subset N(T)$ which is projected onto W_2 under the natural projections $N(T) \to N(T)/T$. Let us consider the group algebra $\mathbb{C}\tilde{W}_2$. It is generated by the delta functions $\delta_w, w \in \tilde{W}_2$. Therefore, for every element $w \in \tilde{W}_2$ we can construct a linear functional on the algebra of regular functions $\mathbb{C}[G]$ such that $f \mapsto f(w), \Delta w = w \otimes w$ where Δ is the coproduct in the dual algebra $\mathbb{C}[G]^*$. There is also a minimal Hopf algebra $\widetilde{U(\mathscr{G})}$ which contains $\mathbb{C}\tilde{W}_2$ and $U(\mathscr{G})$.

We see that $w \in \mathbb{C}[G]^*$ and $\Delta(w) = w \otimes w$, where Δ is the comultiplication in the Hopf algebra $\mathbb{C}[G]^*$. So, the group algebra $\mathbb{C}[\tilde{W}_2]$ defines the Hopf subalgebra of $\mathbb{C}[G]^*$. Also, we can consider the minimal Hopf subalgebra $\widetilde{U(\mathscr{G})} \subset \mathbb{C}[G]^*$ containing both $\mathbb{C}[\tilde{W}_2]$ and the universal enveloping algebra $U(\mathscr{G})$, where $\mathscr{G} =$ Lie G.

0.3. Let G be a simple complex Poisson-Lie group with the Poisson bracket defined by the Yang-Baxter tensor

$$r = \frac{1}{2} \sum_{\alpha \in \Delta_+} X_{-\alpha} \wedge X_\alpha$$

and let $\mathbb{C}[G]_q$ be the corresponding quantized algebra of functions on G (see [2], [15], [18]). If $\bar{w}' \in \mathbb{C}[G]_q^*$ is the quantum analogue of the element $w \in \mathbb{C}[G]^*$, then we cannot expect that $\Delta(\bar{w}') = \bar{w}' \otimes \bar{w}'$. Indeed, being is true, then $\bar{w}'(fg) = \bar{w}'(f)\bar{w}'(g)$, for every $f, g \in \mathbb{C}[G]_q$. The latter equality implies that Hamiltonian structure on G is degenerated at $w \in \tilde{W}_2$ (for proof one passes to limit as $q \to 1$), but this structure is non-denegenerate at any point $w \neq 1$. In the next section we shall try to understand what $\Delta(\bar{w}')$ should be.

0.4. We fix w and consider homomorphism $f(g) \mapsto f(g, w)$ of algebra $\mathbb{C}[G]$. The shift $g \mapsto gw$ is not an automorphism of the Poisson-Lie group G, so $f(g) \mapsto f(gw)$ is not an endomorphism of Poisson-Hopf algebra $\mathbb{C}[G]$. Nevertheless one can introduce in G new Poisson structure such that $g \mapsto gw$ is the morphism of Poisson manifolds (but not of Poisson-Lie group morphism). For example, if $w = w_0$ is the element of maximal length, then the new bracket is defined by $\{f_1, f_2\}_+ = r^{MV}(\partial_M f_1 \partial_V f_2 + \partial'_M f_1 \partial'_V f_2)$ where $\{f_1, f_2\} = r^{MV}(\partial_M f_1 \partial_V f_2 - \partial'_M f_1 \partial'_V f_2)$ is initial bracket (notations are as in [3]). To understand the quantum analogue of the morphism $g \mapsto gw$ we define the latter in terms of 1-cocycle $\varphi : \mathscr{G} \to \Lambda^2 \mathscr{G}, \varphi(X) = [r, \Delta_0(X)]$, where $\Delta_0(x) = x \otimes 1 + 1 \otimes x$ (see [3]).

We restrict ourselves to the case $w = w_0$. Here the morphism $g \mapsto gw_0$ corresponds to the substitution $\varphi_{w_0}(x) = \varphi(x) + 2\Delta_0(x) r$ for $\varphi(x)$. It is easy to see that the quantum analogue for $\varphi_{w_0}(x)$ is the new comultiplication in $U_h(\mathscr{G}) : \Delta_{w_0}(x) = \Delta(x) R$, where R is the universal R-matrix for quantized universal enveloping algebra $U_h(\mathscr{G})$. Indeed,

$$\Delta_{w_0}(x) - \Delta'_{w_0}(x) = h(\varphi(x) + 2\Delta_0(x) r) + O(h^2)$$

where Δ'_{w_0} is the opposite comultiplication. The comultiplication Δ_{w_0} does not define the Hopf algebra structure in $U_h(\mathscr{G})$, but it is coassociative because of the conditions $(\Delta \otimes id)(R) = R^{13}R^{23}$, $(id \otimes \Delta)(R) = R^{12}R^{13}$ (see (1.14) below).

We see that the quantum analogue of the morphism $g \mapsto g, w_0$ is the coalgebra morphism $(U_h(\mathscr{G}), \Delta) \to (U_h(\mathscr{G}), \Delta_{w_0})$, $a \mapsto a\bar{w}'_0$.

It follows that $\Delta(a)\Delta(\bar{w}'_0)R = \Delta(a)\bar{w}'_0 \otimes \bar{w}'_0$ i.e.

$$\Delta(\bar{w}'_0) = R^{-1}(\bar{w}'_0 \otimes \bar{w}'_0) . \tag{0.1}$$

Similar hypothetical formula can be obtained for any $\bar{w}'$ the quantum analogue of element $w \in W$. For example, $\Delta(\bar{S}'_i) = R_i^{-1}(\bar{S}'_i \otimes \bar{S}'_i)$ if S_i is the simple reflection and R_i is the universal R-matrix for corresponding $sl(2)_i$-triples (see section 2 below).

0.4.1. REMARK. One can obtain another motivation for (0.1) from the equality $(Ad_{w_0} \otimes Ad_{w_0})\varphi(x) = -\varphi\left(Ad_{w_0}(x)\right)$ for every $x \in \mathscr{G}$. Hence $Ad_{\bar{w}'_0}$ must be an antiautomorphism of coalgebra $U_h(\mathscr{G})$ and (1.13) below follows from (0.1).

0.5. Let us discuss the existence problem for $\bar{w}' \in \mathbb{C}[G]^*_q$. In [16] for the case $G = SL_2(\mathbb{C})$ the quantum element $\bar{w}_0$ was constructed as the Gelfand-Naimark-Segal state corresponding to some irreducible representation of the algebra $\mathbb{C}[SL_2(\mathbb{C})]_q$. It should be noted however that $\bar{w}'_0$ in (0.1) differs from $\bar{w}_0$ defined above by a factor that belongs to quantized Cartan universal enveloping algebra. Later in [15], [17] $\bar{w}$ was defined in a a general case for any element w of the Weyl group. This definition gives Coxeter relations for quantized simple reflectiàons $\bar{S}_i$ and $\bar{S}_j$ (see [15], [17]). The relations $\bar{S}_i^2 = 1$ does not hold, but there is some substitute for it (see [15], [16], [6], [17] and section 2 below). It is possible to introduce the quantum analogue for the quantum Hopf algebra $\widetilde{U(\mathscr{G})}$, and it's impossible for the Hopf algebra $\mathbb{C}[\tilde{W}_2]$ (see [15] and section 2).

0.6. Here we discuss the relation between quantum Weyl group and universal R-matrices. The general construction for quantum R-matrix of quantized simple Lie algebras is given in [2], section 13. As far as we know, the explicit formula is obtained in the case $sl(n)$ only ($n=2$ in [2], $n \geq 2$ in [14]). The hypothetical formula (0.1) allows us to hope for a formula for R of the following kind. Let $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ be the reduced expression. Then (0.1) gives

$$(0.2) \qquad R = Ad\bar{s}_{i_1}\bar{s}_{i_2}\dots\bar{s}_{i_{N-1}}(\tilde{R}_{i_N})\dots Ad_{\bar{s}_{i1}}(\tilde{R}_{i_2})\tilde{R}_{i_1}e^{\frac{h}{2}t_0}$$

where $Ad_{\bar{s}_i}(x) = \bar{s}_i x \bar{s}_i^{-1}$ and $\tilde{R}_i$, $t_0 = \sum_k I_k \otimes I_k$ are defined in (2.3), (3.1) below.

It should be noted that formula (0.2) and other similar formulas are closely related to the order in $f\ \Delta_+$ introduced in section 2.1 below. For applications of this order to factorizations problems see [7].

We shall show in this work that formula of the type (0.2) really holds. Unfortunately, we are still unable to obtain the formulae of the type (0.1) immediately. Therefore, we are to use the quantum double construction of [2], section 13.

Nevertheless, the quantum Weyl group is actively used throughout our work.

0.7. The element $\bar{w}_0'$ from (0.1) has a series of interesting properties. One can show that $(\bar{w}_0')^2$ lies in the centre of $\mathbb{C}[G]_q^*$. Furthermore, one can easily derive

$$\Delta\left((\bar{w}_0')^2\right)\left((\bar{w}_0')^2 \otimes (\bar{w}_0')^{-2}\right) = \left(R^{21}R^{12}\right)^{-1}$$

where $R^{12} = R$, $R^{21} = \sigma(R)$, $\sigma(x \otimes y) = y \otimes x$. Hence, $(\bar{w}_0')^2 = e^{-\frac{hC}{2}}$, where $e^{-\frac{hC}{2}}$ is the quantum analogue of the Casimir operator introduced in [4], section 5. The element $\bar{w}_0'$ plays an important role in construction of the ribbon Hopf algebras and knot invariants (see [13]).

0.8. Most part of our paper is related to R-matrix. Also we give one simple application of quantum Weyl group to Hecke algebras in section 4. More precisely we prove that if $L(\Lambda)$ is a q-analogue of the adjoint representation of $\mathcal{G}$ and $L(\Lambda)_0 \subset L(\Lambda)$ is the space of the zero-weight vectors, than the quantum Weyl group acts on $L(\Lambda)_0$ as (generalised) Hecke algebra.

After preparation of this work we received preprint [10] in which P.-B.-W. basis in $U_q(\mathcal{G})$ was constructed in a way similar to ours. We note that the automorphisms T_w of [10] is equal to $Ad_{\bar{w}^{-1}}$, up to normalization (see [17]). Previously A-D-E case was considered in [9].

We also received a letter from A.N. Kirillow. He communicated that in his joint paper with N. Yu. Reshetikhin the multiplicative formula for universal R-matrix is obtained.

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1. THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS

1.1. Let $\mathscr{G}$ be a simple complex Lie algebra. We fix:

a) the Cartan decomposition $\mathscr{G} = n_+ \oplus \mathscr{H} \oplus n_-$ where $n_\pm$ are nilpotent subalgebras, $\mathscr{H}$ is Cartan subalgebra;

b) the decomposition $\Delta = \Delta_+ \cup \Delta_-$ of the root system corresponding to a); here $\Delta_+(\Delta_-)$ is the set of positive (negative) roots;

c) the root basis $\{\alpha_1, \dots \alpha_r\}$ of Δ_+;

d) the invariant scalar product $(,)$ in $\mathscr{H}^*$.

We denote by W the Weyl group generated by reflection $s_i : \Lambda \mapsto \Lambda - \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$, $\lambda \in \mathscr{H}^*$, $1 \le i \le r$.

1.2. Let $U_h(\mathscr{G})$ be $\mathbb{C}[[h]]$-algebra generated (in h-adic sense, i.e. as an algebra complete in the h-adic topology) by $\{H_i, X_i^+, X_i^-\}_{i=1}^r$ with defining relations

$$[H_i, H_j] = 0, \quad [H_i, X_j^\pm] = \pm(\alpha_i, \alpha_j) X_j^\pm \tag{1.1}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{sh\left(\frac{h}{2} H_i\right)}{sh \frac{h}{2}} \tag{1.2}$$

$$\sum_{k=0}^{n_{ij}} (-1)^k \binom{n_{ij}}{k}_{q_i^2} q_i^{-k(n_{ij}-k)} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{n_{ij}-k} = 0 \tag{1.3}$$

where $n_{ij} = 1 - a_{ij}$, $\left((a_{ij})\right)$ is the Cartan matrix, $\binom{n}{k}_t = \frac{(n)_t!}{(k)_t!(n-k)_t!}$, $(n)_t! = (1)_t \dots (n)_t$, $(n)_t = \frac{t^n-1}{t-1}$, $q_i = q^{(\alpha_i,\alpha_i)/2}$, $q = e^{h/2}$.

We can introduce the structure of Hopf algebra in $U_h(\mathscr{G})$ if we define comultiplication, conity and antipode by

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_1, \quad \Delta(X_i^\pm) = X_i^\pm \otimes e^{\frac{h}{y} H_i} + e^{-\frac{h}{y} H_i} \otimes X_i^\pm \tag{1.4}$$

$$\varepsilon(H_i) = \varepsilon(X_i^\pm) = 0 \tag{1.5}$$

$$S(H_i) = -H_i, \quad S(X_i^\pm) = -q_i^{\pm 1} X_i^\pm . \tag{1.6}$$

The complete subalgebras of $U_h(\mathscr{G})$ generated by $\{H_i, X_i^+\}_{i=1}^r$ and by $\{H_i, X_i^-\}_{i=1}^r$ are denoted by $U_h(\mathscr{B}_+)$ and $U_h(\mathscr{B}_-)$ respectively.

1.2.1. We shall consider also the Hopf algebra over $\mathbb{C}$ defined for fixed $q > 1$. It can be defined as subalgebra in $U_h(\mathscr{G})$ generated by $\{X_i^\pm, q^{\pm \frac{H_i}{2}}\}_{i=1}^r$.

Also, we can define $U_q(\mathscr{G})$ by the relations

$$(1.7)\qquad q^{\pm\frac{H_i}{2}}q^{\pm\frac{H_j}{2}} = q^{\pm\frac{H_j}{2}}q^{\pm\frac{H_i}{2}},\, q^{\pm\frac{H_i}{2}}q^{\mp\frac{H_j}{2}} = q^{\mp\frac{H_j}{2}}q^{\pm\frac{H_i}{2}},\, q^{\frac{H_i}{2}}q^{-\frac{H_i}{2}} = 1$$

$$(1.8)\qquad q^{\frac{H_i}{2}}X_j^{\pm} = q^{\pm\frac{(\alpha_i,\alpha_j)}{2}}X_j^{\pm}, q^{\frac{H_i}{2}}, q^{-\frac{H_i}{2}}X_j^{\pm} = q^{\pm\frac{(\alpha_i,\alpha_j)}{2}}X_j^{\pm}q^{-\frac{H_i}{2}}$$

$$(1.9)\qquad [X_i^+, X_j^-] = \delta_{ij}\frac{q^{H_i}-q^{-H_i}}{q-q^{-1}}$$

and by relation (1.3). Comultiplication, counity and antipode are given by

$$(1.10)\qquad \Delta\left(q^{\pm\frac{H_i}{2}}\right) = q^{\pm\frac{H_i}{2}}\otimes q^{\pm\frac{H_i}{2}},\ \Delta(X_i^{\pm}) = X_i^{\pm}\otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}}\otimes X_i^{\pm}$$

$$(1.11)\qquad \varepsilon\left(q^{\pm\frac{H_i}{2}}\right) = 1,\quad \varepsilon(X_i^{\pm}) = 0$$

$$(1.12)\qquad S\left(q^{\pm\frac{H_i}{2}}\right) = q^{\mp\frac{H_i}{2}},\ S(X_i^{\pm}) = -q_i^{\pm 1}X_i^{\pm}.$$

In our considerations the Hopf algebra $U_q(\mathscr{G})$ plays a subordinate role. We introduce it primarily, to show that some results on quantum Weyl group are valid for fixed $q > 1$ (the latter condition can be relaxed).

1.3. Let us remind the scheme for construction of universal R-matrix for algebra $U_h(\mathscr{G})$ ([2], section 13). First one constructs the so-called quantum double $\mathscr{D}(U_h(\mathscr{E}_+))$ of Hopf albgera $U_h(\mathscr{E}_+)$. As linear space, $\mathscr{D}(U_h(\mathscr{E}_+))$ is $U_h(\mathscr{E}_+)\otimes U_h(\mathscr{E}_+)^*$ to be precise, the former is some completion of the latter). The Hopf algebra structure in $\mathscr{D}(U_h(\mathscr{E}_+))$ is uniquely defined by following conditions:

a) the embedding $U_h(\mathscr{E}_+)\hookrightarrow\mathscr{D}(U_h(\mathscr{E}_+))$ is Hopf algebra homomorphism;

b) the embedding $U_h(\mathscr{E}_+)^*\hookrightarrow\mathscr{D}(U_h(\mathscr{E}_+))$ is algebra homomorphism and coalgebra antihomorphism;

c) the canonical element $\bar{R}\in\mathscr{D}(U_h(\mathscr{E}_+))\otimes\mathscr{D}(U_h(\mathscr{E}_+))^*$ defines quasi triangular Hopf algebra structure in $\mathscr{D}(U_h(\mathscr{E}_+))$ i.e. $\bar{R}$ satisfies the following equations

$$(1.13)\qquad \Delta'(a) = \bar{R}\Delta(a)\bar{R}^{-1},\quad a\in\mathscr{D}(U_h(\mathscr{E}_+))$$

$$(1.14)\qquad (\Delta\otimes id)(\bar{R}) = \bar{R}^{13}\bar{R}^{23},\quad (id\otimes\Delta)(\bar{R}) = \bar{R}^{13}\bar{R}^{12}$$

where $\Delta' = \sigma_0\Delta$, $\sigma(x\otimes y) = y\otimes x$, $\bar{R}^{12} = \bar{R}\otimes 1$ and so on. Then one proves that there exists a Hopf algebra isomorphism $U_h(\mathscr{E}_+)^0\simeq U_h(\mathscr{E})$, where A^0 is Hopf algebra A with opposite comultiplication: $\Delta_{A^0} = \Delta'_A$. Hence, there exists a Hopf algebra epimorphism $\pi:\mathscr{D}(U_h(\mathscr{E}))\to U_h(\mathscr{G})$ and $R = (\pi\otimes\pi)(\bar{R})$ defines quasi triangular Hopf algebra structure in $U_h(\mathscr{G})$.

1.4. Let M be the rigid quasi tensor category (see [12]) consisting of finite dimensional representations ρ of $U_q(\mathscr{G})$ such that the spectrum of the endomorphisms $\rho\left(q^{\frac{H_i}{2}}\right)$ consists of positive numbers only. The Hopf algebra that consists of matrix elements of representations of the category M is denoted by $\mathbb{C}[G]_q$ and is called the algebra of regular functions on the quantum group G.

1.4.1. EXAMPLE. We fix the vertex i of Dynkin diagram of the Lie algebra $\mathscr{G}$ and denote by $U_{q_i}(sl(2))$ the subalgebra of $U_q(\mathscr{G})$ generated by $X_i^{\pm}, q^{\pm\frac{H_i}{2}}$.

The corresponding algebra of regular functions on quantum group $SL_2(\mathbb{C})$ is denoted by $\mathbb{C}[SL_2(\mathbb{C})]_{q_i}$. It is a $*$-Hopf algebra, as it is of the type $\mathbb{C}[G]_q$ (see [20], [18]). Its irreducible $*$-representations are described in [16].

1.4.2. We recall the definition of quantum Weyl group ([15]). In [16], section 4, the quantized element $w_0^{(q_i)}$ is certain Gelfand-Naimark-Segal state on the $*$-algebra $\mathbb{C}[SL_2(\mathbb{C})]_{q_i}$ (see [16], section 4.2). Let $\varphi_i : U_{q_i}(sl(2)) \to U_q(\mathscr{G})$ be the embedding of Hopf algebras and let $\varphi_i^* : \mathbb{C}[G]_q \to \mathbb{C}[SL_2(\mathbb{C})]_{q_i}$ be the corresponding epimorphism of quantized algebras of functions. Let $s_i \in W$ corresponds to i-vertex of Dynkin diagram. The quantized simple reflection $\bar{s}_i \in \mathbb{C}[G]_q^*$ is defined by

$$\bar{s}_i(f) = w_0^{(q_i)}\left(\varphi_i^*(f)\right) \tag{1.15}$$

(see [15]).

1.4.3. THEOREM. ([15]) *Let $i \neq j$ and let m_{ij} be the order of the element $s_i s_j$ in the usual Weyl group. Then $\bar{s}_i\bar{s}_j\bar{s}_i\ldots = \bar{s}_j\bar{s}_i\bar{s}_j\ldots$ (m_{ij} factors in both sides).* ∎

1.4.4. REMARK. One can consider the rigid quasi tensor category, isomorphic to M, generated by indecomposable finite dimensional $U_h(\mathscr{G})$-modules (see [4], section 4). It allows us to define elements $\tilde{s}_i \in h, \mathbb{C}[G]_h^*$ similar to $\bar{s}_i$. Here $h, \mathbb{C}[G]_h^*$ is an Hopf algebra over $\mathbb{C}[[h]]$ constructed as $\mathbb{C}[G]_q^*$. Every where below we work with $U_h(\mathscr{G})$ and denote $\tilde{s}_i$ as $\bar{s}_i$. The results of the sections 1.4.8, 1.4.10, 2.2 and 2.4.2 hold for fixed $q > 1$ also.

1.4.5. We introduce the new generators $E_i = X_i^+ q^{\frac{-H_i}{2}}$, $F_i = X_i^- q^{\frac{H_i}{2}}$ in $U_h(\mathscr{G})$.

1.4.6. PROPOSITION. ([2]) *The universal R-matrix for $U_h(sl(2)_i)$ is*

$$R_i = \left(\sum_{n\geq 0} \frac{(1-q_i^{-2})^n}{(n)_{q_i^{-2}}} E_i^n \otimes F_i^n\right) q^{\frac{H_i\otimes H_i}{(\alpha_i,\alpha_i)}} i$$

∎

1.4.7. PROPOSITION. ([2]) ([15], [6], [17]) *Let* $\bar{s}'_i = q^{\frac{-H_i^2}{2(\alpha_i,\alpha_i)}}\bar{s}_i$. *Then*

a) $\Delta(\bar{s}'_i) = R_i^{-1}(\bar{s}'_i \otimes \bar{s}'_i)$

b) $\varepsilon(\bar{s}'_i) = 1$

c) $S(\bar{s}'_i) = (\bar{s}'_i)^{-1}\sum_{n\geq 0}\frac{(1-q_i^{-2})^n}{(n)_{q_i^{-2}}!}E_i^n q^{-\frac{H_i^2}{(\alpha_i,\alpha_i)}} S(F_i^n)$

d) $(\bar{s}'_i)^2 = e^{-\frac{hC_i}{2}}$, *where* $e^{-\frac{hC_i}{2}}$ *is the quantum analogue of Casimir element of* $U_h(sl(2)_i)$ *(see [4], section 4).*

1.4.8. PROPOSITION. ([16]) *We have in* $\mathbb{C}[G]_h^*$: $\bar{s}_i E_i \bar{s}_i^{-1} = -q^{-H_i}F_i$, $\bar{s}_i F_i \bar{s}_i^{-1} = -E_i q^{H_i}$. ∎

1.4.9. REMARK. The proposition 1.4.6 - 1.4.8 allow us to define the minimal Hopf subalgebra $\widetilde{U}_h(\mathcal{G}) \subset \mathbb{C}[G]_h^*$ which contains $U_h(\mathcal{G})$ and elements $\bar{s}_i, \bar{s}_i^{-1}$. For $\mathcal{G} = sl(2)$, this is done in [16], [6] and the general case is considered in [15]. This $\widetilde{U_h(s)}$ is called the quantum Weyl group.

1.4.10. Let A be an Hopf algebra with comultiplication Δ and antipode S. We define the $A \otimes A$-module structure by

$$(a \otimes \mathscr{E}) \cdot x = axS(\mathscr{E}) \tag{1.16}$$

(see [4]). The adjoint action of A in A is defined by $ad_a(x) = \Delta(a)\cdot x$. Let A^0 be the Hopf algebra which coincides with A as algebra and has opposite comultiplication Δ'. We set $ad'_a(x) = \Delta'(a)\cdot x$ and obtain another adjoint action of A in A, A^0 being A as algebra. Below we use these notations in the case $A = U_h(\mathcal{G})$.

PROPOSITION. *For* $i \neq j$:

$$\bar{s}_i E_j \bar{s}_i^{-1} = c_{ij}^+(q)\, ad_{E_i}^{-a_{ij}}(E_j) \tag{1.17}$$

$$\bar{s}_i F_j \bar{s}_i^{-1} = c_{ij}^-(q)(ad'_{F_i})(ad'_{F_i})^{-a_{ij}}(F_j) \tag{1.18}$$

where

$$c_{ij}^{\pm}(q) = \frac{(-1)^{a_{ij}}q^{(3-2\varepsilon_{ij}^{\pm})(\alpha_i,\alpha_j)/2}}{\sqrt{(-1)^{a_{ij}}\prod_{k=0}^{a_{ij}-1}\left(\sum_{m=0}^{k}[(\alpha_i,\alpha_j)+m(\alpha_i,\alpha_j)]_q\right)}}$$

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}, \quad \varepsilon_{ij}^{\pm} = \begin{cases} 0 & for + \\ 1 & for - \end{cases}.$$

Let us sketch the proof of (1.17) (the proof of (1.18) is similar). We consider the vector space

$$V_{ij} = \text{span}\ \{E_j, ad_{E_i}(E_j), \dots, ad_{E_i}^{-a_{ij}}(E_j)\}\ .$$

It is clear that V_{ij} is a $U_{q_i}(sl(2))$ -module.

E_j being the lowest weight vector and $ad_{E_i}^{-a_{ij}}(E_j)$ being the highest weight vector, we have $ad_{\bar{s}_i'}(E_j) = c_{ij}^{+}(q) ad_{E_i}^{-a_{ij}}(E_j)$ (see [16], section 4). In order to compute $c_{ij}^{+}(q)$ we use proposition 1.4.7 and obtain

$$ad_{\bar{s}_i'}(E_j) = \Delta(\bar{s}_i') \cdot E_j = \left(R_i^{-1} \cdot (\bar{s}_i' E_j (\bar{s}_i')^{-1})\right)(Ri \circ 1)\ .$$

Hence $R_i \cdot \left(ad_{\bar{s}_i'}(E_i)\right) = \left(\bar{s}_i' E_j (\bar{s}_i')^{-1}\right)(R_i \cdot 1)$ and to complete the proof it suffices to use the following two equalities

$$R_i \cdot 1 = \sum_{h \geq 0} \frac{(1 - q_i^{-2})^n}{(n)_{q_i^{-2}}!} E_i^n q^{-\frac{H_i^2}{(\alpha_i,\alpha_i)}} S(F_i^n)$$

$$E_i ad_{E_i}^{-a_{ij}}(E_j) = q^{(\alpha_i,\alpha_j)} ad_{E_i}^{-a_{ij}}(E_j) E_i\ .$$

■

1.4.11. REMARK. If $\mathcal{G} = sl(n)$, then the automorphism $x \mapsto \bar{S}_i x \bar{S}_i^{-1}$ and the automorphism T_i' in [9] coindide (see [17]).

2.THE ROOT VECTORS OF $U_h(\mathcal{G})$

2.1. Let $w_0 \in W$ be the element of maximal length and let $w_0 = s_{i_1} \dots s_{i_N}$ be any reduced expression. Then, we set $\bar{w}_0 = \bar{s}_{i_1} \dots \bar{s}_{i_N}$ and note that $\bar{w}_0$ does not depend on the choice of reduced expression due to 1.4.3. Let us consider the following set of roots

$$\mathcal{D} = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, \dots s_{i_1} \dots s_{i_{N-1}}\alpha_{i_N}\}\ .$$

It is well known (see, for example, proposition 17 and corollary 2 or ([1]) that $\mathcal{D} = \Delta_+$. The set $\mathcal{D}$ can be ordered from right to left. We introduce in Δ_+ the induced order. Let $\alpha = s_{i_1} \dots s_{i_{p-1}} \alpha_{i_p}$. We define root vectors by induction:

(2.1) $$E_\alpha = T_{i_1} \dots T_{i_{p-1}}(E_{i_p}), \quad F_\alpha = T_{i_1} \dots T_{i_{p-1}}(F_{i_p})$$

where $T_j(x) = \bar{s}_j x (\bar{s}_j)^{-1}$.

One can prove that the definition (2.1) is correct (as in [8] one reduces to the case of rank 2 group).

2.2.1. REMARK. The special choice of the reduced expression for w_0 allows us to simplify the right-hand side in (2.2) (see [5] for the case $\mathcal{G} = sl(n)$ and [8] for the A-D-E case).

2.3. Let $\alpha : [1,N] \to \Delta_+$ be the order described in 2.1. We have the following Poincaré -Birkhoff-Witt theorem.

2.3.1. THEOREM. *The set of monomials*

$$\{E_{\alpha(N)}^{k_N} \dots E_{\alpha(1)}^{k_1} H_1^{m_1} \dots H_r^{m_r}\}_{m_i,k_i \in \mathbf{Z}_+}$$

form a basis of $U_h(\mathcal{B}_+)$ *and the set of monomials*

$$\{E_{\alpha(N)}^{k_N} \dots E_{\alpha(1)}^{k_1} H_1^{m_1} \dots H_r^{m_r} F_{\alpha(N)}^{eN} \dots F_{\alpha(1)}^{l_1}\}_{m_i,k_i l_i \in \mathbf{Z}_+}$$

form a basis of $U_h(\mathcal{G})$.

2.3.2. REMARK. The analogue of theorem 2.3.1 holds for $U_q(\mathcal{G})$. One has to substitute $q^{\frac{H_i}{2}}$ for H_i and to allow $m_i \in \mathbf{Z}$.

2.4. Here we state some resutls about action of comultiplication on root vectors. We restrict ourselves the the case $U_h(\mathcal{B}_+)$.

We note that $\Delta(E_i) = E_i \otimes 1 + q^{-H_i} \otimes E_i$ and, due to 1.4.7, that

$$\Delta(\bar{s}_i) = \tilde{R}_i^{-1}(\bar{s}_i \otimes \bar{s}_i) \tag{2.3}$$

where

$$\tilde{R}_i = \sum_{n \geq 0} q^{-n^2(\alpha_i,\alpha_i)} \frac{(1-q_i^{-2})^n}{(n)_{q_i}^{-2}!} E_i^n q^{-nH_i} \otimes F_i^n q^{nH_i} .$$

For every $\alpha = s_{i_1} \dots s_{i_{k-1}} \alpha_{i_k}$ we set

$$\tilde{R}_\alpha = T_{i_1} \dots T_{i_{k-1}}(\tilde{R}_{i_k}), \ \tilde{\mathbf{R}}_{>\beta} = \prod_{\alpha > \beta} \tilde{R}_\alpha$$

the product being taken according to our order.

2.4.1. PROPOSITION. *Let H_β correspond to the root β under the canonical isomorphism $\mathscr{G} \tilde{\to} \mathscr{G}^*$.*

Then

$$\Delta(E_\beta) = \tilde{R}^{-1}_{>\beta}(E_\beta \otimes 1 + q^{H\beta} \otimes E_\beta)^n \tilde{R}_{>\beta}. \tag{2.4}$$

The proof follows from (2.3) and definitions. ∎

2.4.2. Let $U_h(n_+)_{\alpha(k)}$ be the subspace of $U_h(n_+)$ with basis $\{E^{m_{k-1}}_{\alpha(k-1)} \cdots E^{m_1}_{\alpha(1)}\}_{m_j \in \mathbb{Z}_+}$. One can prove the following theorem by means of (2.2) and (2.4.1).

THEOREM. a) $\Delta(E^n_\beta) - (E_\beta \otimes 1 + q^{H\beta} \otimes E_\beta)^n \in U_h(n_+)_\beta \otimes U_h(\mathscr{B}_+)$

b) $U_h(n_+)_\beta$ *is a subalgebra of* $U_h(n_+)$;

c) $\Delta\left(U_h(n_+)_\beta\right) \subset U_h(n_+)_\beta \otimes U_h(\mathscr{B}_+)$. ∎

2.4.3. In the case $\mathscr{G} = sl(n)$ theorem 2.4.2 was proved in [14].

3. THE ALGEBRA $U_h(\mathscr{B}_+)^*$ AND THE QUANTUM R-matrix.

3.1. Let us introduce the linear forms $\xi_1, \ldots, \xi_r, \eta_\alpha, \alpha \in \Delta_+$, defined by: $\xi_i(H_i) = 1$, zero on other monomials; $\eta_\alpha(E_\alpha) = 1$, zero on other monomials, and let $\eta_i = \eta_{\alpha_i}$. It is easy to prove the following

PROPOSITION. a) $[\xi_i, \eta_j] = -\delta_{ij}\frac{h}{2}\eta_j$

b) $[\xi_i, \xi_j] = 0$

c) $ad^{-a_{ij}}_{\eta_i}(\eta_j) = 0$, $i \neq j$

d) $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i, \Delta(\eta_i) = \eta_i \otimes 1 + e^{\sum_{k=1}^r(\alpha_i, \alpha_k)\xi_k} \otimes \eta_i$. ∎

This proposition yields the Hopf algebra isomorphism $U_h(\mathscr{B}_+)^0 \tilde{\to} U_h(\mathscr{B})$ such that $\eta_j \mapsto (1 - q_j^{-2})F_j$, $\exp\left(\sum_{k=1}^r(\alpha_i, \alpha_k)\xi_k\right) \mapsto q^{H_i}$.

3.2. PROPOSITION. *For any* $K_j, P_j, K'_j, P'_j \in \mathbb{Z}_+ \langle \eta^{K_N}_{\alpha(N)} \cdots \eta^{K_1}_{\alpha(1)} \xi_1^{p_1} \cdots K_N \cdots \xi_r^{p_r}, E^{K'_N}_{\alpha(N)} \cdots E^{K_i}_{\alpha(1)} H_1^{P'_1} \cdots H_r^{P'_r} \rangle = \prod_{i=1}^r \delta_{p_i, p'_i} \prod_{j=1}^N \delta_{k_j, k'_j} \prod_{i=1}^r (p_i)!, \prod_{j=1}^N (K_j)_{q_j^{-2}}!$ *where* $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$.

Proof. We use the induction and 2.2, 2.4.1, 2.4.2: at every step, the only thing to do is the pairing with

$$(E_\beta \otimes 1 + q^{-H\beta} \otimes E_\beta)^n .$$

■

3.3. Let Φ be the isomorphism in 3.1. Propositions 3.1, 3.2 give $\Phi(\eta_\alpha) = (1 - q_\alpha^{-2}) F_\alpha$. Thys, recollecting 1.3, we obtain

$$R = \sum_{\substack{k_j \in \mathbb{Z}_+ \\ m_j \in \mathbb{Z}_+}} \left(m_1! \dots m_r! (k_1)_{q^{-2}_{\alpha(1)}}! \dots (k_N)_{q^{-2}_{\alpha(1)}} \right)^{-1} \times$$

$$\times E_{\alpha(N)}^{k_N} \dots E_{\alpha(1)}^{k_N} H_1^{m_1} \dots H_r^{m_r} \otimes \tag{3.1}$$

$$\Phi \left(\eta_{\alpha(N)}^{k_N} \dots \eta_{\alpha(1)}^{k_1} \dots \xi_1^{m_1} \dots \xi_r^{m_r} \right) =$$

$$= \prod_{\alpha \in \delta_+} \exp_{q_\alpha^{-2}} \left((1 - q_\alpha^{-2}) E_\alpha \otimes F_\alpha \right) q^{t_0}$$

where $t_0 = \sum_k I_k \otimes I_k$, $\{I_k\}_{k=1}^r$ is the orthonormal basis in $\mathscr{S}$ and $\exp_t(x) = \sum_{n \geq 0} \frac{x^n}{(n)_t!}$, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$.

3.4. Let $\bar{w}_0' = q^{-\frac{1}{2} \sum_{k=1}^r I_k^2} \bar{w}_0$. Our results give the following (cf. (0.1):

$$\Delta(\bar{w}_0') = R^{-1}(\bar{w}_0' \otimes \bar{w}_0') . \tag{3.2}$$

For $\mathscr{G} = sl(n)$, (3.2) was proved in [17]. It seems very interesting to find the proof without using the explicit formula for R-matrix, since we were able to derive (3.1) from (3.2).

For $sl(n)$ formula (3.1) was proved in [14].

4. RELATION WITH HECKE ALGEBRAS

4.1. We recall the definition of an Hecke algebra (see [1]). Let $q \in \mathbb{C}$ and let $\mathscr{G}$ be a simply-laced Lie algebra such that $(\alpha_i, \alpha_i) = 2$, for every i. An Hecke algebra $H_q(W)$ is an associative unital algebra over $\mathbb{C}$ generated by $\{T_i\}_{i=1}^r$ and defining relations

$$T_i T_j T_i \dots = T_j T_i T_j \dots \quad (m_{ij} \text{ factors in both sides}) \tag{4.1}$$

$$(T_i - q^{-2})(T_i + 1) = 0 \tag{4.2}$$

where m_{ij} is defined in theorem 1.4.3.

4.2. Let $L(\Lambda)$ be a finite-dimensional $U_h(\mathscr{G})$-module (see [4]) with the highest weight Λ such that $\Lambda(H_i) = \alpha_{\max}(H_i)$, where $\alpha_{\max}$ is the maximal root. Therefore $L(\Lambda)$ is the quantum analogue of the adjoint representation of $\mathscr{G}$. Set $L(\Lambda)_0 = \{x \in L(\Lambda) | ax = 0$ for every $a \in \mathscr{G}\}$.

4.2.1. THEOREM. *For every* $i \in [1, r]$, *we have*

a) $\bar{s}_i(L(\Lambda)_0) \subset L(\Lambda)_0$;

b) $(\bar{s}_i - q_i^{-2})(\bar{s}_i + 1)|_{L(\Lambda)_0} = 0, q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$.

Proof (1). The first statement is evident. To prove b) we fix $i \in [1, r]$ and consider $L(\Lambda)$ as $U_h(sl(2)_i)$ module. It is well known that $L(\Lambda) = \oplus_k V_k^{(i)}$, where every $V_k^{(i)}$ is a «quantum deformation» of the simple $sl(2)$-module $\bar{V}_k^{(i)}$ such that $\dim \bar{V}_k^{(i)} \leq 3$ (actually, this is true in the case $q = 1$).

Therefore we can use the results of [16], where eigenvalues of $\bar{s}_i$ were computed for any simple $U_h(sl(2)_i)$-module and finish the proof.

4.3. We conclude that the quantum Weyl group acts on $L(\Lambda)_0$ as «generalized Hecke algebra» (we remind that in a general case $q_i \neq q_j$ for $i \neq j$). Therefore, if $\mathscr{G}$ is a simply-laced Lie algebra (such as in 4.1), then the corresponding quantum Weyl group acts in $L(\Lambda)_0$ as Hecke algebra $H_q(W)$, $q = e^{\frac{h}{2}}$. This result is closely related to section 1 of the paper [9] by G. Lusztig. Thus one can prove that the automorphisms $T_i : L(\Lambda) \to L(\Lambda)$ from [9] coincide with $\bar{s}_i : L(\Lambda) \to L(\Lambda)$ and we reconstruct the result of [11], section 1, by means of theorem 4.2.

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